# Linearity throughout Mathematics

A Big Idea Beyond just Algebra 1

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# Why Big Ideas?

- U.S. math curriculum is “a mile wide and an inch deep.”
- We cannot continue to teach too many topics in not enough depth.
- Need to focus on important mathematics
- Develop big ideas systematically across the curriculum and across the grades

# Why Linearity?

- Fundamental idea
- College profs: Give us students who understand linearity
- Helps with nonlinear also:
  Key theme – linear approximating nonlinear

# Overview
## Lines and Linearity in:

Algebra

Geometry

Statistics

Calculus

Linear Algebra and others

# Lines and Linearity in Algebra

- Solve linear equations, emphasize multiple connected representations
- Graph linear functions, with a twist
- Recursive view

# Lines and Linearity in Geometry

- The story of parallel lines
  –> Non-Euclidean geometries
- Lines in drawing
- Line segments approximate curves
- Linear transformations, e.g., rotations, and matrix representation

# Lines and Linearity in Statistics

- Linear regression
- Linearizing data

# Lines and Linearity in Calculus, Lin Alg, and …

- Derivative as linear approximation
- Vector spaces (linear spaces)
- Linear transformations
- Locally linear: $R^n$

# Resources

- NCTM E-Standards Website
- NCTM Illuminations Website
- NCTM Navigations
- Core-Plus Mathematics
- Iowa’s Every Student Counts project

# Start with the basics …

Solve: $3x + 4 = 10$

(10 seconds)

# Solve: $3x + 4 = 10$

Think of as many other ways to solve as you can, discuss with your neighbors.

(Don’t solve, just discuss other possible solution strategies.)

(2 minutes)

# Solve: $3x + 4 = 10$

Anyone solve by first dividing by 3?

Might a student do it that way?

Is it an appropriate strategy?

A teachable moment?

Always deal with the constant term 1st?

What about – Solve: $1/3\ x + 4 = 10/3$

# Moral:

- Teach for understanding.
- Teach for *deep procedural knowledge.* (flexibility, judge appropriateness, comprehension, apply, evaluate)
- Multiple representations and multiple solution strategies – see connections and evaluate for appropriateness

<u>Example</u>:

"Solving Linear Equations" in packet

Source: Iowa's *Every Student Counts* Project

# Solve $3x + 4 = 10$

- Each member of group solves in one of five ways.
- Work together with group to explain and analyze different methods:
  -- How and why it works
  -- Advantages
  -- Disadvantages
- Summarize

# Varying Parameters … with a twist

Source: NCTM *E-Standards* Electronic Example:

“Exploring Linear Functions: Representational Relationships”

http://my.nctm.org/standards/document/eexamples/chap7/7.5/index.htm

# Link the parameters, then vary

Comes from student exploration.

Pattern?

Prove it? (Later -- homework)

Value:

- Students do mathematics
- Explore, look for patterns, explain, prove
- Learn about lines, algebraic symbolic representations, algebraic reasoning

# So far …

- Solve linear equations
- Graph linear functions, note slope and y-intercept

Value Added:

- multiple connected representations and solution strategies
- linked parameters, algebraic proof

# Iterate Linear Functions

Source: *Core-Plus Mathematics* (Glencoe)

"Lesson 3: Iterating Fucntions"

In packet

# Iterating Linear Functions

Launch:
together, p. 1

Explore:
In groups at your tables, pages 2-3.
**Work together, explain solution to each other before moving to next problem .**

Summarize:
together, after 15 mins.

# Graphical Iteration

- Graphing calculator:
  Seq mode, Web format (more homework)
- Demo on Illuminations website …

# Long-term behavior and slope

- |slope|>1 ==> repelling fixed point
- |slope|<1 ==> attracting fixed point
- Slope=1 ==> no fixed point (y-int$\neq$0)
- Slope=-1 ==> two-cycle
- Pos slope ==> “stairstep” pattern
- Neg slope ==> oscillate (“cobweb”) (graphical iteration is interesting and illustrative -- more homework)

# Regression Line & Correlation

Sources:

NCTM Illuminations

NCTM E-Standards Electronic Example

NCTM Navigations

Iowa’s Every Student Counts Project

(see packet for details)

# Fit Lines to Data - How and Why

1. Examine the Dietary Change data
   **Discuss anything interesting.**
   **Patterns? Trends?**
2. Make predictions from data?
   Suppose you know your cholesterol level now, then you go on diet. What will your cholesterol level be after?
   **Discuss how you could use the data to answer this question.**

# *Think about* a scatterplot (Before, After)

- Any trends in the plot?
- Higher “Before” ==> Higher “After”?
  How would this trend appear in the plot?
- “Before” generally higher or lower than “After”?
  How would this trend appear relative to y=x line?
- 250 changed to 202, about a 20% decrease.
  How would plot look if this always true?

# Find best-fitting line (generic data)

- Create plot using Illuminations applet
- How would you, or your students, fit a line to data?
- How would you, or your students, measure the “error”?
- How to graphically represent error?
- Examine methods with E-Standards applet

# Regression and Correlation

First, what is r, properties of r?

Now, investigate:

- Linear trend ==> High r ?
- High r ==> Linear trend ?
- Cloud of data ==> $r \approx 0$ ?
- $r \approx 0$ == Cloud of data ?

(NCTM Illuminations Activity)

# Regression and Correlation

Now, investigate:

- Linear trend ==> High r  YES
- High r ==> Linear trend  NO
- Cloud of data ==> $r \approx 0$  YES
- $r \approx 0$ ==> Cloud of data  NO

# Moral

- Look at the data, don’t just crunch numbers
- r is a valuable tool for studying the linear association between 2 variables, but it does not fully explain the association (no statistic does)

# Recursive View

Source: NCTM *Navigating through Discrete Mathematics in Grades 9-12*

“A Recursive View of Some Common Functions”

Do parts of this task….

(Note: Recursive view already seen in iterating functions)

# Linear

- Constant rate of change
- NEXT = NOW + 3  (recursive formula)
- $y = 3x + 4$  (explicit formula)

Which representation shows the constant rate of change more intuitively?

# Exponential

- NEXT = NOW * 3
- $y = 4*3^x$

Which is more intuitive for students?

<u>Compare/Contrast with Linear</u>:

- Multiply by 3 vs. add 3 at each step
- Constant multiplier vs. constant add
- Which is constant rate of change?

# Lines & Linearity in Geometry

- <u>"Line" as a fundamental object</u> in synthetic, metric, coordinate, transformation, and vector geometry
- <u>Linear transformations</u>, e.g., rotation, and matrix representation
- <u>Sketching</u> - e.g., perspective drawing Which lines in the actual object remain parallel in the drawing?
- <u>Parallel lines story</u> --> Non-Euclidean Geom

# Lines & Hyperbolic Geometry

“Crocheting Curves: Shaping the Hyperbolic Plane”

Daina Taimina crochet models exhibit

Currently at Carleton College, MN


Seeing is Believing

# Linear Approximates Nonlinear

Differential Calculus: Local Linearity

- Secant lines --> tangent line
- Zoom in, find linear

Geometry and Topology

- Line segments approximate curves
- Manifold is a space that looks locally like (is homeomorphic to) $R^n$, which is linear

# Approximate nonlinear with linear: zoom in

Graph $y = x^3 - x^2 + x$

standard window

Now zoom in at origin

What does graph look like?

# Linearizing Data

Source:

Core-Plus Mathematics (Glencoe)

Activity:

“Straightening Functions”

# Straightening Functions

Do 2-3

4-5, if time

Discuss:

2e, 3e

# Straightening Functions

Original function: $y = \pi x^2$ $(x > 0)$

**Graph of (x, y) is linear? NO, quadratic**

Transformation: $y^* = \sqrt{y}$

Result: $y^* = (\sqrt{\pi})\, x$

**Graph of (x, y*) is linear? YES**

Note: The linearizing function (square root) uses inverse operation, although is not quite the inverse function.

# Straightening Functions

<u>Original function</u>: $y = 2^x$

**Graph of (x, y) is linear? NO, exp.**

<u>Transformation</u>: $y^* = \log y$

Result: $y^* = \log 2^x = x (\log 2)$

**Graph of (x, y*) is linear? YES**

Note: The linearizing function (log) uses the inverse operation, although is not quite the inverse function.

# Straightening Functions Summary

- How would you check if an exponential function is a reasonable fit to a set of data points (x, y)?
- How would you check if a quadratic power function is a reasonable fit to a set of data points (x, y)?
- What is the relationship between a function and its linearizing function?

# Straightening Functions
# Interesting Application

**How does your calculator fit exponential functions to data?**

- Start with (x, y) data (you think it's exponential)
- Log the y's
- Now consider (x, log y) data
- Fit a line (least squares), get: $y^* = ax + b$
- That is, $\log y = ax + b$
- Solve for y, get: $y = 10^{(ax + b)} = \ldots =$ exponential

Note: The exponential function is <u>not</u> necessarily the the one with least squared error.

# Summary: Linearity

<u>Algebra</u>: Linearity is a basic *relationship*

<u>Geom</u>: Linearity is a basic pattern of *shape*

<u>Stat</u>: Linearity is a basic pattern of *data*

<u>Calculus</u>: Linearity is a basic pattern of *change*

<u>Linear Algebra</u>: Everything is linear.

<u>Topology(manifolds</u>): Everything is locally linear

<u>Key Theme</u>: Linear approximating nonlinear

**Develop this big idea across the grades and across the strands of mathematics.**